A Generalized Algorithm for Learning Positive and Negative Grammars with Unconventional String Models

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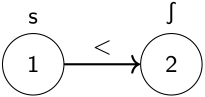

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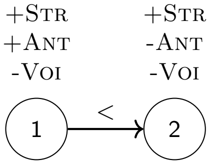

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  - What about collections of **allowed** combinations?
- **This talk:** algorithm for learning grammars as collections of **allowed** or **forbidden** feature-based combinations in a **unified** way

Example: Samala Sibilant Harmony

Examples (Samala Sibilant Harmony)

- Subsequences such as [s...s] that agree in $\pm$ANTERIOR are allowed
- Subsequences such as [s...ʃ] which disagree are banned

✓ [hasxintilawas] ✗ [hasxintilawaʃ]

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Negative Grammar (G^{-})

$[+\textsc{Ant}][\text{-}\textsc{Ant}] \in G^{-} \Rightarrow$ sʃt $\notin L(G^{-})$

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Positive Grammar (G^{+})

$[+\textsc{Str}][\text{-}\textsc{Str}],[+\textsc{Ant}][\text{-}\textsc{Ant}] \in G^{+}$
$\Rightarrow$ sʃt $\in L(G^{+})$

Since $[+\textsc{Ant}][\text{-}\textsc{Ant}]$ covers [sʃ] and $[+\textsc{Str}][\text{-}\textsc{Str}]$ covers [ʃt]

Why Positive Grammars?

Psycholinguistic Motivation

Computational Motivation

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By **fixing** k — the size of the learned substructures — we can straightforwardly adapt the algorithm of Chandlee et al. (2019) to learn the **most general positive and negative grammars** over **feature-based models**

Table of Contents

Table of Contents

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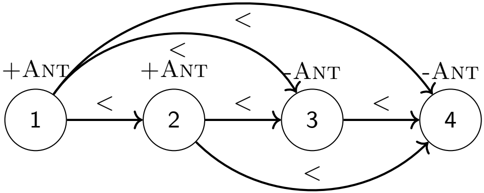
Size $|S|$ of an R-structure = cardinality of its domain

Precedence and Successor Models

Precedence Model: $M^{<}(w) := \langle D^w; <, [R^w_\sigma]_{\sigma\in\Sigma}\rangle$

- $D^w = \{1, ..., |w|\}$ is the **domain** of positions in w
- $<:= \{(i,j) \in D^w \times D^w \mid i < j\}$ is the **general precedence** relation



(Büchi, 1960; McNaughton and Papert, 1971; Rogers et al., 2013)

Precedence and Successor Models

Successor Model: $M^{\lhd}(w) \coloneqq \langle D^w; \lhd, [R^w_\sigma]_{\sigma \in \Sigma} \rangle$

- $D^w = \{1, ..., |w|\}$ is the **domain** of positions in w
- $\lhd \coloneqq \{(i, i+1) \in D^w \times D^w\}$ is the **successor** relation

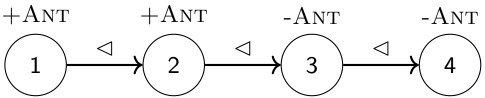


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Table of Contents

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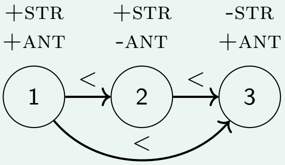

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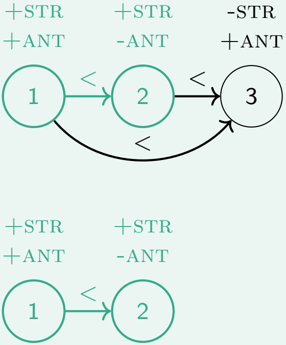

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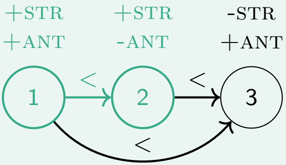

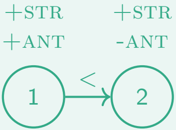

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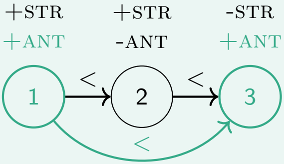

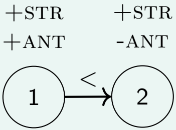

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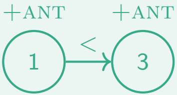

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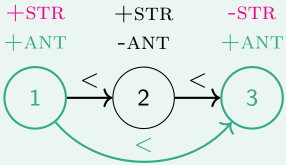

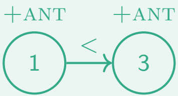

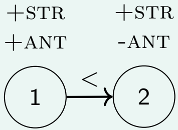

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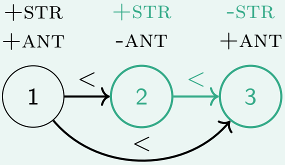

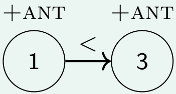

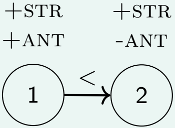

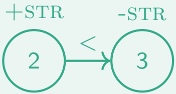

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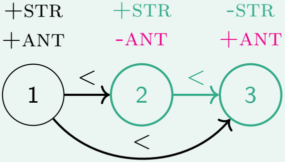

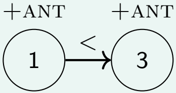

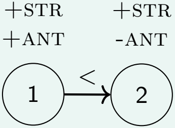

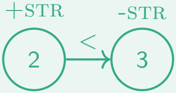

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k-Subfactors and k-Maxfactors

Definition: k-Subfactors

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Let the set of k-subfactors of an R-structure B be given by:

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Table of Contents

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Language defined by G depends on its **interpretation**:

Negative Grammar (G^-)

Elements of G^- are forbidden, and strings in $L(G^-)$ contain no forbidden subfactors

Positive Grammar (G^+)

Elements of G^+ are permissible, and strings in $L(G^+)$ are those which are *tiled* by these elements

Positive Grammars: Tiling

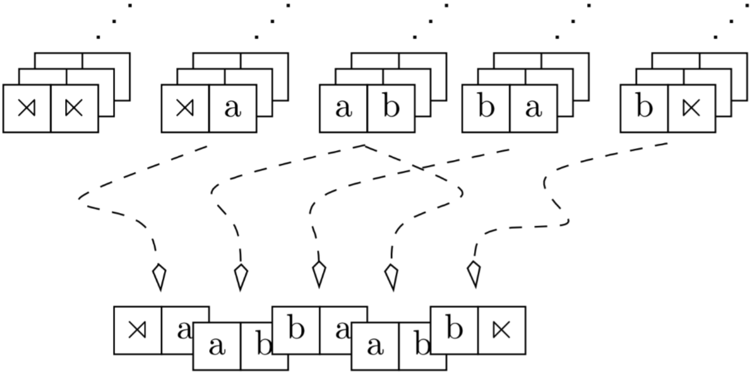

Figure courtesy of Rogers and Heinz (2014)

Languages of Positive vs. Negative Grammars

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The language $L(G^-)$ of G^- is given by:

$$L(G^-) = \{w \in \Sigma^* \mid (\forall S \in \textsc{MFac}_k(M, w)) \; [\textsc{SFac}_k(S) \cap G^- = \emptyset]\}$$

or equivalently by:

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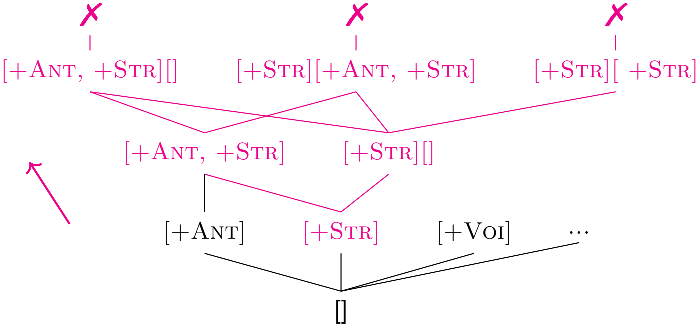
	$\in G$	$\notin G$
$\forall$	**Positive Grammar**	**Negative Grammar**
$\exists$	**Negative Grammar**	**Positive Grammar**

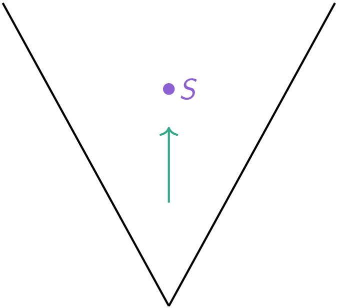
Table of Contents

Previous Work

Crucial insight of **Chandlee et al. (2019)**: **grammatical entailment**

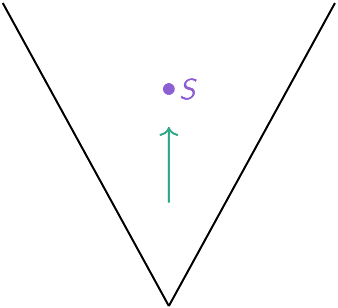


Chandlee et al. (2019) Algorithm



- **Bottom-up** traversal

Chandlee et al. (2019) Algorithm



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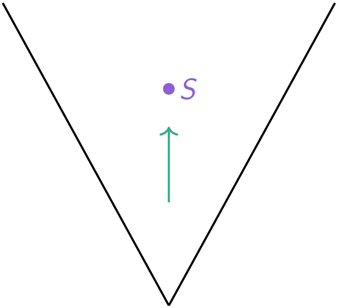

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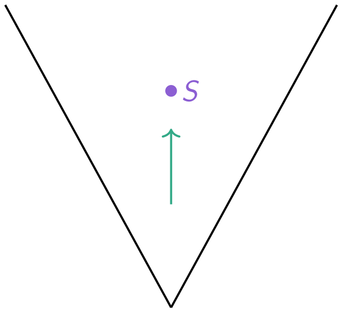

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  - If not, **posit a constraint:** $S \in G^-$
  - If so, **cannot posit a constraint**
    Add the **least superfactors** of S to the queue to be considered next

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Least Superfactors of S ($\texttt{NextSupFact}(S)$) are the superfactors of S that differ minimally from S

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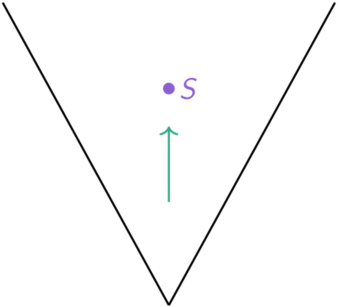
Examples (Next superfactors of $[+\textsc{Ant}]$)

If $S = [+\textsc{Ant}]$ and the only features available are $\pm\textsc{Ant}$, $\pm\textsc{Voi}$ and $\pm\textsc{Str}$:

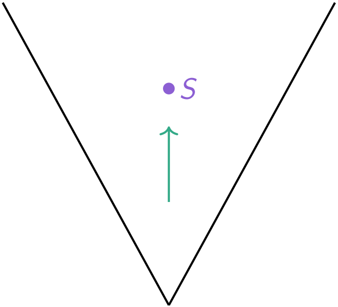
$$\begin{aligned}\texttt{NextSupFact}(S) = \{&[+\textsc{Ant},\ \text{-}\textsc{Str}], [+\textsc{Ant},\ +\textsc{Str}]\\ &[+\textsc{Ant},\ \text{-}\textsc{Voi}], [+\textsc{Ant},\ +\textsc{Voi}]\}\end{aligned}$$

Intuition



For a **negative grammar**:

- Add S to G^- if $S \not\sqsubseteq x$ for any $x \in D$



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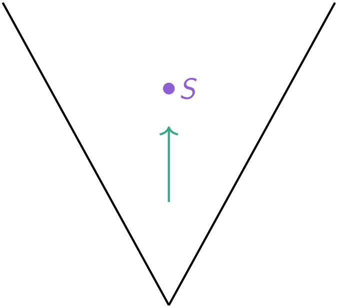
Intuition



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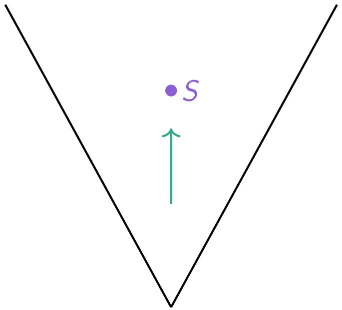

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Extensions

Extensions of S ($\textsc{Ext}_k(S)$) are all k-maxfactors that are superfactors of S

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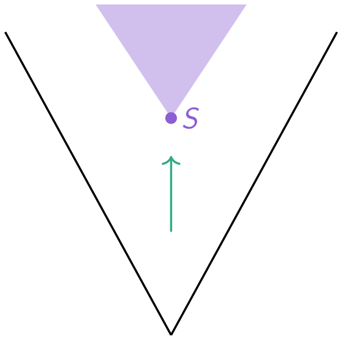
Examples (Extension of $[+\textsc{Ant}]$)

If $S = [+\textsc{Ant}]$ and the only features available are $\pm\textsc{Ant}$, $\pm\textsc{Voi}$ and $\pm\textsc{Str}$:

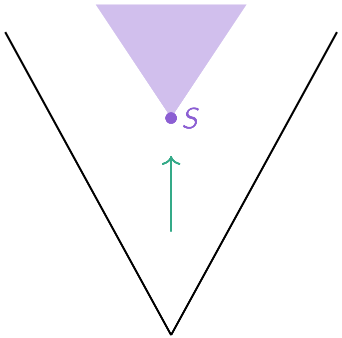
$$\begin{aligned}\textsc{Ext}_k(S) = \{&[+\textsc{Ant},\ \text{-}\textsc{Str},\ +\textsc{Voi}], [+\textsc{Ant},\ +\textsc{Str},\ +\textsc{Voi}]\\ &[+\textsc{Ant},\ \text{-}\textsc{Str},\ \text{-}\textsc{Voi}], [+\textsc{Ant},\ +\textsc{Str},\ \text{-}\textsc{Voi}]\}\end{aligned}$$

Learning Algorithm



- **Bottom-up** traversal

Learning Algorithm



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- For a given subfactor S, check whether:

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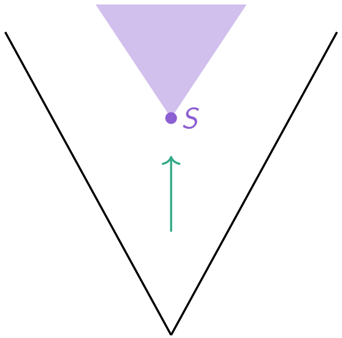

- **Bottom-up** traversal
- For a given subfactor S, check whether:
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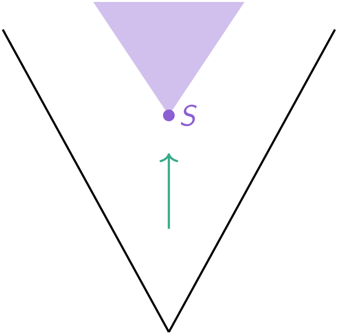
$$(\forall s' \in \textsc{Ext}_k(S))[\nexists x \in D, s' \leq x])$$

Learning Algorithm



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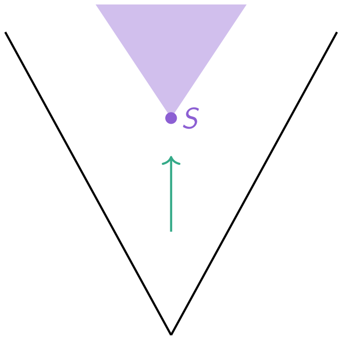

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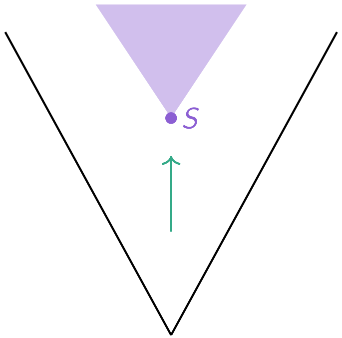

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Learning Guarantees

Fix Σ, model M, positive integer k, and polarity p. For any language $L \in \mathscr{L}^p(M, k)$ and for any finite sample $D \subseteq L$, return a grammar G^p such that:

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3. G^p includes R-structures S that are restrictions of R-structures S' in other grammars G' that also satisfy (1) and (2). That is, for all G' satisfying (1) and (2) and for all $S' \in G'$, there exists some $S \in G^p$ such that $S \sqsubseteq S'$.

Table of Contents

Applying the Algorithm: Samala Sibilant Harmony

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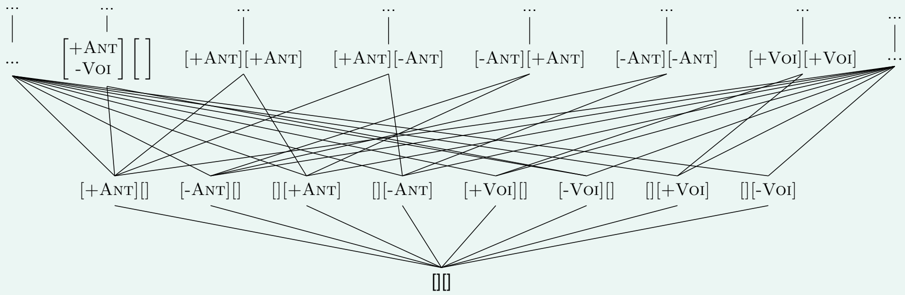

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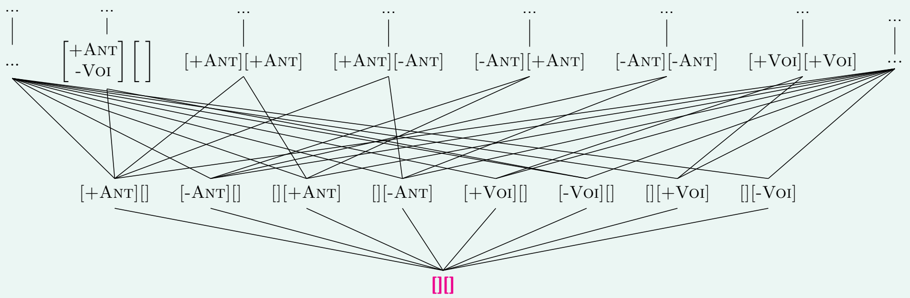

Negative Grammar

Positive Grammar

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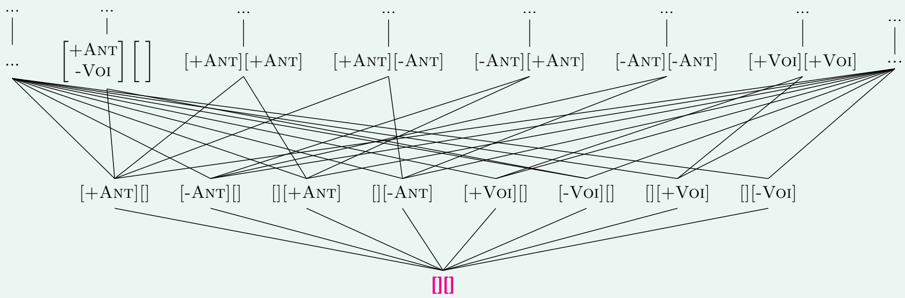

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Is there any element in $\text{Ext}_k([][])$ which **is** a 2-maxfactor of some $x \in D$?

Positive Grammar

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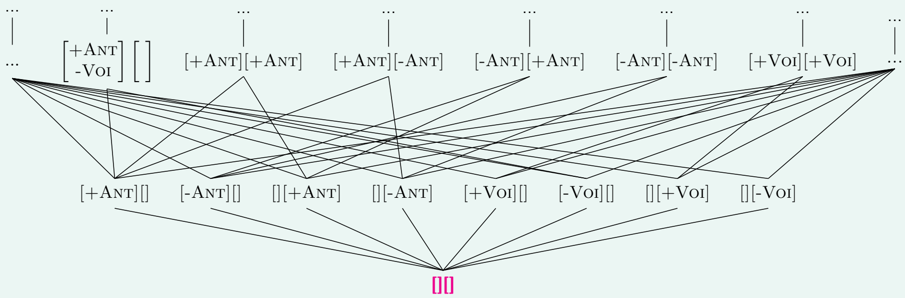

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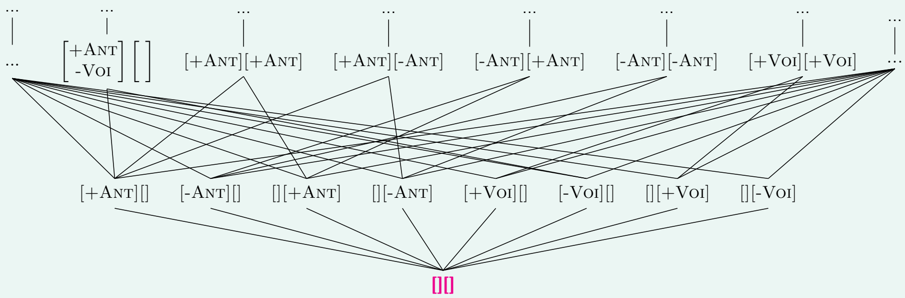

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e.g. $[+\textsc{Voi}, +\textsc{Ant}][+\textsc{Voi}, +\textsc{Ant}] \in \textsc{Ext}_k([][])$, but [z...z] is **licit and attested**.

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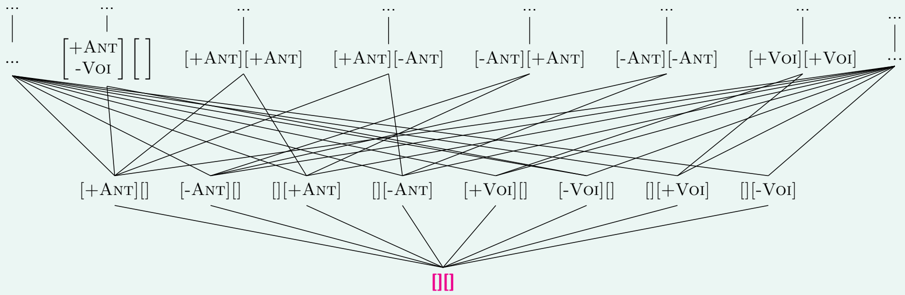

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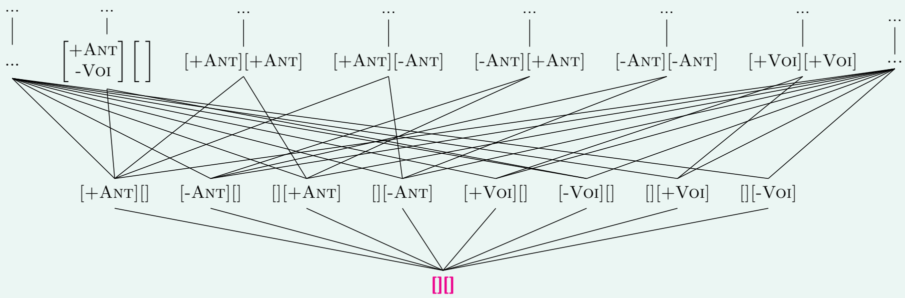

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Keep searching!

Applying the Algorithm: Samala Sibilant Harmony

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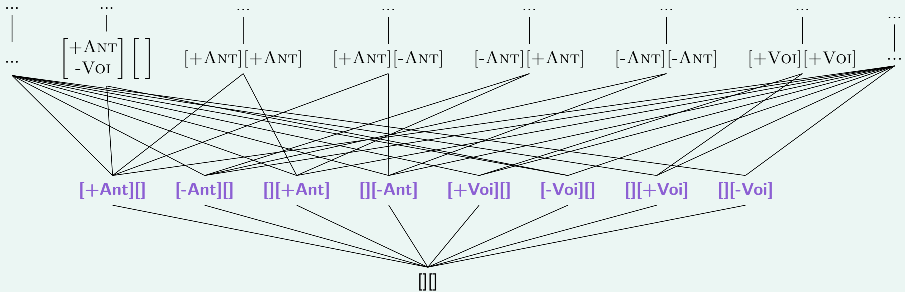

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Applying the Algorithm: Samala Sibilant Harmony

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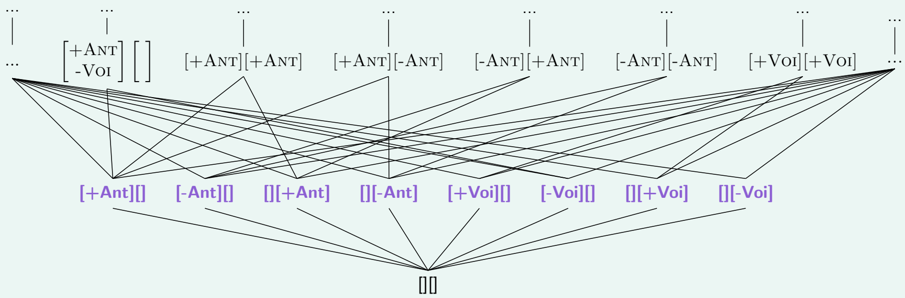

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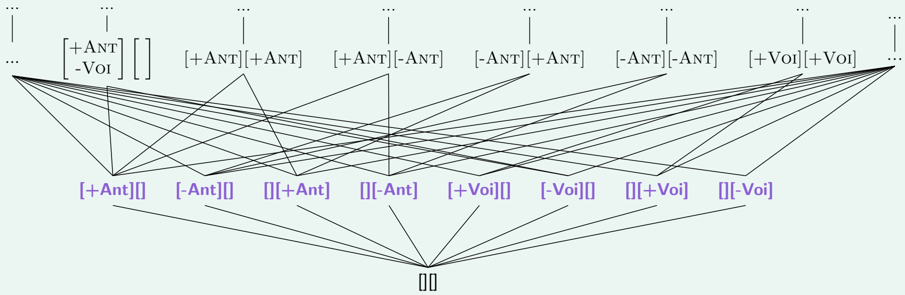

- Extract the **least superfactors** of [][] and consider each of them
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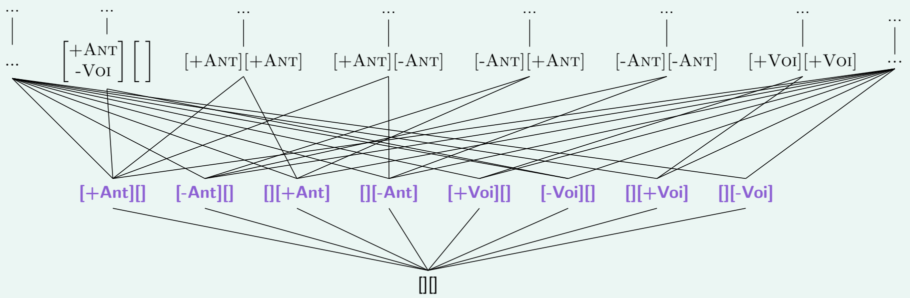

- Extract the **least superfactors** of [][] and consider each of them
- Still not specified enough:
  - Any subfactor with ±Ant specified in one position has licit and illicit maxfactors in its extension
    [+Ant][] $\sqsubseteq$ [+Ant][+Ant] but [+Ant][] $\sqsubseteq$ [+Ant][-Ant]

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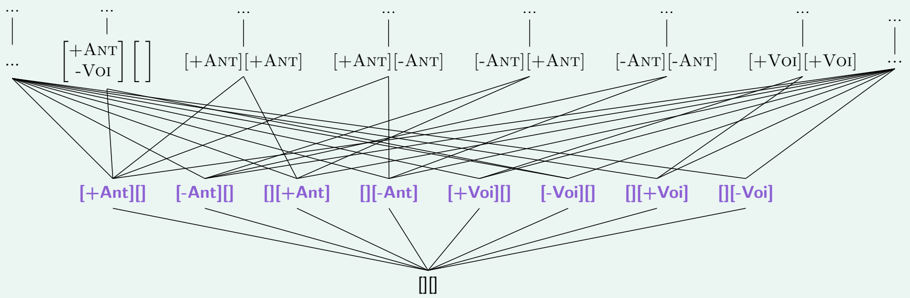

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    [+ANT][] $\sqsubseteq$ [+ANT][+ANT] but [+ANT][] $\sqsubseteq$ [+ANT][-ANT]
  - $\pm$VOI has no bearing on licitness

Applying the Algorithm: Samala Sibilant Harmony

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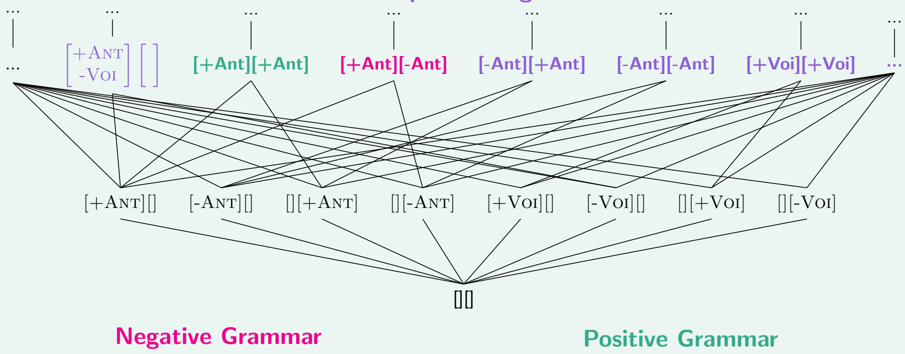
Keep searching!



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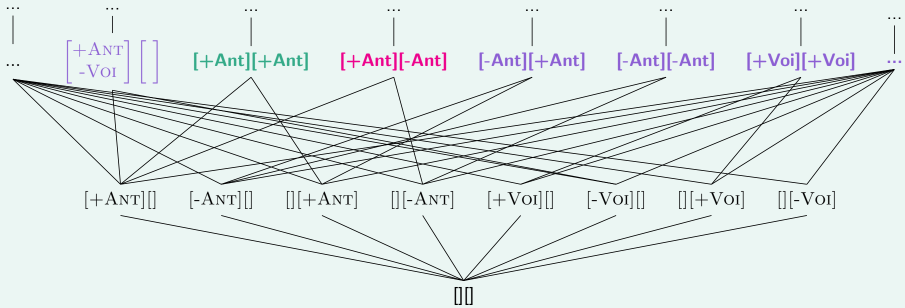

Negative Grammar

Is there any element in $\textsc{Ext}_k([+\textsc{Ant}][-\textsc{Ant}])$ which **is** a 2-maxfactor of some $x \in D$? **No!**

Positive Grammar

Applying the Algorithm: Samala Sibilant Harmony

Examples (Samala Sibilant Harmony)

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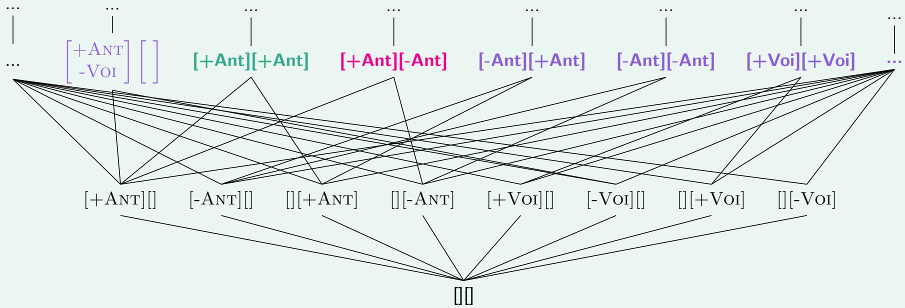

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Applying the Algorithm: Samala Sibilant Harmony

Examples (Samala Sibilant Harmony)

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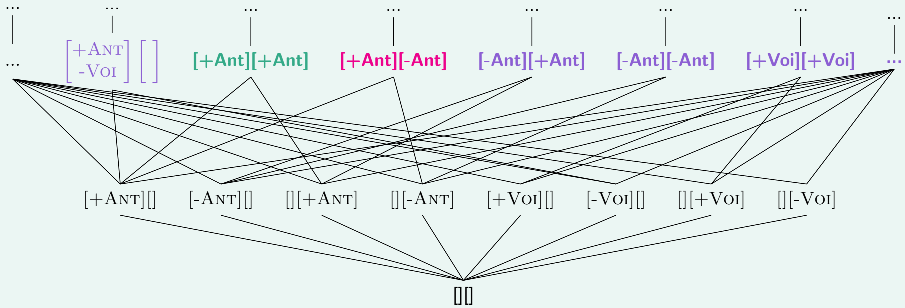


Negative Grammar

Is there any element in $\textsc{Ext}_k$([+ANT][-ANT]) which **is** a 2-maxfactor of some $x \in D$? **No!**

[+ANT][-ANT] is added to G^-

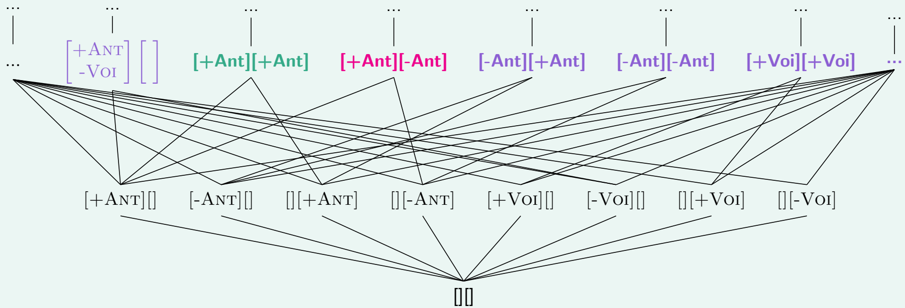
Positive Grammar

Is there any element in $\textsc{Ext}_k$([+ANT][+ANT]) which **is not** a 2-maxfactor of some $x \in D$? **No!**

[+ANT][+ANT] is added to G^+

Applying the Algorithm: Samala Sibilant Harmony

Examples (Samala Sibilant Harmony)



Negative Grammar

We may later reach [+Ant][-Ant, +Voi] but we won't consider it.

[+Ant][-Ant] being banned entails [+Ant][-Ant, +Voi] being banned

Positive Grammar

We may later reach [+Ant][+Ant, +Voi] but we won't consider it.

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  - As G^+ grows, $L(G^+)$ **grows**
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  - Initially, G^+ allows **nothing**, while G^- allows **everything**

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    - Within a single search of the hypothesis space
    - When applied to incrementally larger data sets as a proxy for incremental learning

Thank you!

I am grateful to Jeff Heinz, Thomas Graf, Jon Rawski, Logan Swanson, and the SCiL reviewers for discussion.

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Restrictions

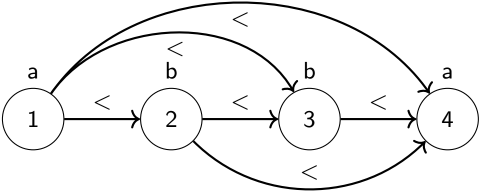
Definition: Restriction

An R-structure A is a **restriction** of an R-structure B if $D^A \subseteq D^B$ and for each m-ary relation R_i in the model signature:

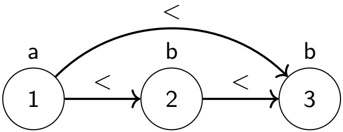
$$R_i^A = \{(x_1, ..., x_m) \in R_i^B \mid x_1, ..., x_m \in D^A\} \quad (1)$$

Intuition: identify a subset D^A of the domain of B and retain only those relations in B whose elements are wholly within D^A

R-structure B



R-structure A



$D^A = \{1, 2, 3\} \subset D^B = \{1, 2, 3, 4\}$

Subfactor

Definition: Subfactor

An R-structure A is a **subfactor** of an R-structure B (notated $A \sqsubseteq B$) if there exists a restriction B' of B and a bijection h such that for all $R_i \in R$, if $R_i(x_1, ..., x_m)$ holds in A, then $R_i(h(x_1), ..., h(x_m))$ holds in B'.

Intuition: A is a subfactor of B if there is a mapping between D^A and some subset of D^B and all relations that hold in A also hold over the corresponding elements in B

Maxfactor

Definition: Maxfactor

An R-structure A is a **maxfactor** of an R-structure B (notated $A \leq B$) iff $A \sqsubseteq B$ and for each m-ary relation R_i, whenever $R_i(x_1, ..., x_m)$ holds in B, $R_i(h^{-1}(x_1), ..., h^{-1}(x_m))$ holds in A.

Intuition: A is a maxfactor of B if $A \sqsubseteq B$ and and all relations that hold in B also hold over the corresponding elements in A

Extensions

Extensions of a Subfactor

The extensions of a subfactor S are defined as follows:

$$\begin{aligned}\textsc{Ext}_k(S) = &\{A \in \textsc{Sfac}_k(M, \Sigma^*) \mid \\ &S \sqsubseteq A \wedge (\nexists A')[|A'| = k \wedge A \sqsubseteq A']\}\end{aligned} \tag{2}$$

Intuition: extensions of S are all k-maxfactors that are superfactors of S.

Examples (Extension of $[+\textsc{Ant}]$)

If $S = [+\textsc{Ant}]$ and the only features available are $\pm\textsc{Ant}$, $\pm\textsc{Voi}$ and $\pm\textsc{Str}$:

$$\begin{aligned}\textsc{Ext}_k(S) = &\{[+\textsc{Ant},\ \text{-}\textsc{Str},\ +\textsc{Voi}], [+\textsc{Ant},\ +\textsc{Str},\ +\textsc{Voi}] \\ &[+\textsc{Ant},\ \text{-}\textsc{Str},\ \text{-}\textsc{Voi}], [+\textsc{Ant},\ +\textsc{Str},\ \text{-}\textsc{Voi}]\}\end{aligned}$$

Next Superfactor

Next Superfactor

We extract the more specific superfactors of S by calling `NextSupFact`(s) where `NextSupFact()` is defined as follows:

$$\begin{aligned}\texttt{NextSupFact}(S) = &\{A \in \textsc{SFac}_k(M, \Sigma^*) \mid \\ &S \sqsubseteq A \wedge (\nexists A')[S \sqsubseteq A' \sqsubseteq A]\}\end{aligned} \tag{3}$$

Intuition: `NextSupFact()` returns the *least* superfactors for S.

Examples (Next superfactors of $[+\textsc{Ant}]$)

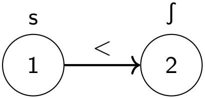
If $S = [+\textsc{Ant}]$ and the only features available are $\pm\textsc{Ant}$, $\pm\textsc{Voi}$ and $\pm\textsc{Str}$:

$$\begin{aligned}\texttt{NextSupFact}(S) = &\{[+\textsc{Ant},\ \text{-}\textsc{Str}], [+\textsc{Ant},\ +\textsc{Str}] \\ &[+\textsc{Ant},\ \text{-}\textsc{Voi}], [+\textsc{Ant},\ +\textsc{Voi}]\}\end{aligned}$$

Conventional vs. Unconventional String Models

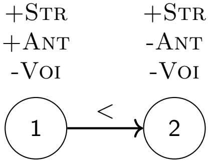
Conventional String Models

- **Mutually-exclusive unary relations** label each domain element with the single property of being some $\sigma \in \Sigma$
- **Segments** in phonological applications



Unconventional String Models

- **Non-exclusive unary relations** allow distinct alphabetic symbols to share properties
- **Features** in phonological applications



(Strother-Garcia et al., 2016; Vu et al., 2018)

Connectedness

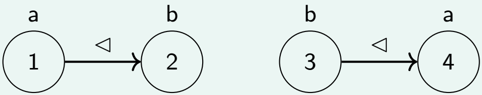
Connectedness

An R-structure $S = \langle D; R_1, R_2, ..., R_n \rangle$ is connected iff $(\forall x, y \in D)[(x, y) \in C^*]$, where C^* is defined as the symmetric transitive closure of:

$$\begin{aligned} C = &\{(x, y) \in D \times D \mid \\ &\quad \exists i \in \{1...n\}, \exists (x_1...x_m) \in R_i \\ &\quad \exists s, t \in \{1...m\}, x = x_s, y = x_t\} \end{aligned}$$

Intuition: domain elements x and y of S belong to C if they belong to some non-unary relation R_i in S

Examples (Disconnected R-Structure)



The Cost of Interdefinability

For symbolic models, negative & positive grammars are straightforwardly interdefinable:

$$G^+ = \Sigma^k \setminus G^- \quad G^- = \Sigma^k \setminus G^+$$

Two complications for feature-based models:

Number of k-Subfactors

- A model with n binary features defines $s \leq 2^n$ segments
- Segment-based model: no more than $(s)^k \leq (2^n)^k$ k-factors
- Under a feature-based model: $(3^n)^k$ possible k-subfactors since each feature can be positive, negative, or unspecified

Conversion Process

- For symbolic models, need to simply check whether some k-factor $f \in \Sigma^k$ is in G^-
  if $[\text{s}\ldots\text{ʃ}] \in G^-$ then $[\text{s}\ldots\text{ʃ}] \notin G^+$
- For feature-based models, a k-subfactor f should not be added to G^+ if $f \in G^-$, but also if $(\exists g \in G^-)[f \sqsubseteq g \vee g \sqsubseteq f]$.

Some Lemmas

Lemma 1: Maxfactor-Subfactor Containment

Let k be some positive integer and let M be some model of Σ^*. For any $w \in \Sigma^*$ and for any $F \in \textsc{Sfac}_k(M, w)$, we have that:

$$[\exists G \in \textsc{Mfac}_k(M, w)](F \sqsubseteq G)$$

Lemma 2: Union of Subfactors of Maxfactors

Let k be some positive integer and let M be some model of Σ^*. For any $w \in \Sigma^*$, we have that:

$$\bigcup_{S \in \textsc{Mfac}_k(M,w)} \textsc{Sfac}_k(S) = \textsc{Sfac}_k(M, w)$$